Discrete Mathematics with Applications, 5th Edition
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Answers for Test Bank Questions: Chapters 1-4

Please use caution when using these answers. Small differences in wording, notation, or choice of examples or counterexamples may be acceptable.

Chapter 1

1. a. a remainder of 1 when it is divided by 4 and a remainder of 3 when it is divided by 7

   b. an integer n; n is divided by 7 the remainder is 3

2. a. a positive real number; smaller than r

   b. positive real number r; there is a positive real number s

   Fill in the blanks to rewrite the following statement with variables:

3. There is an integer whose reciprocal is also an integer.

4. a. have three sides

   b. has three sides

   c. has three sides

   d. is a triangle; has three sides

   e. T has three sides

5. a. have additive inverses

   b. an additive inverse

   c. y is an additive inverse for x

6. a. less than or equal to every positive integer

   b. positive integer m; less than or equal to every positive integer

   c. less than or equal to n

7. (a) The set of all integers n such that n is a factor of 9.

   Or: The set of all elements n in **Z** such that n is a factor of 9.

   Or: The set of all elements n in the set of all integers such that n is a factor of 9.

   (b) $\{1, 3, 9\}$

8. (a) No

   (b) Yes

   (c) Yes

   (d) No

9. a. $\{(a, u), (a, v), (b, u), (b, v), (c, u), (c, v)\}$

   b. $\{(u, a), (v, a), (u, b), (v, b), (u, c), (v, c)\}$

10. a. Yes; No; No; Yes

    b. $\{(3, 15), (3, 18), (5, 15)\}$

    c. domain is $\{3, 5, 7\}$; co-domain is $\{15, 16, 17, 18\}$.

    d. Draw an arrow diagram for R.

    e. No: R fails both conditions for being a function from A to B. (1) Elements 5 and 7 in A are not related to any elements in B, and (2) there is an element in A, namely 3, that is related to two different elements in B, namely 15 and 18.

11. a. No; Yes; No; Yes

    b. Draw the graph of R in the Cartesian plane.

    c. No: R fails both conditions for being a function from **R** to **R**. (1) There are many elements in **R** that are not related to any element in **R**. For instance, none of 0, 1/2, and -1 is related to any element of **R**. (2) there are elements in **R** that are related to two different elements in **R**. For instance 2 is related to both 1 and -1.

12. a. $G(2) = c$

    b. Draw an arrow diagram for G.

13. $F \neq G$. Note that for every real number x,

$$G(x) = (x-2)^2 - 7 = x^2 - 4x + 4 - 7 = x^2 - 4x - 3,$$

whereas

$$F(x) = (x+1)(x-3) = x^2 - 2x - 3.$$

Thus, for instance,

$$F(1) = (1+1)(1-3) = -4 \quad \text{whereas} \quad G(1) = (1-2)^2 - 7 = -6.$$

Chapter 2

1. e
2. e
3. a. The variable S is not undeclared or the data are not out of order.

   b. The variable S is not undeclared and the data are not out of order.

   c. Al was with Bob on the first, and Al is not innocent.

   d. $-5 > x$ or $x \geq 2$
4. The statement forms are not logically equivalent.

   Truth table:

p	q	$\sim p$	$p \vee q$	$\sim p \wedge q$	$p \vee q \to p$	$p \vee (\sim p \wedge q)$
T	T	F	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	F	T
F	F	T	F	F	T	F

   Explanation: The truth table shows that $p \vee q \to p$ and $p \vee (\sim p \wedge q)$ have different truth values in rows 3 and 4, i.e, when p is false. Therefore $p \vee q \to p$ and $p \vee (\sim p \wedge q)$ are not logically equivalent.

5. *Sample answers:*

   Two statement forms are logically equivalent if, and only if, they always have the same truth values.

   Or: Two statement forms are logically equivalent if, and only if, no matter what statements are substituted in a consistent way for their statement variables the resulting statements have the same truth value.

6. *Solution 1:* The given statements are not logically equivalent. Let p be "Sam bought it at Crown Books," and q be "Sam didn't pay full price." Then the two statements have the following form:

$$p \to q \quad \text{and} \quad p \vee \sim q.$$

   The truth tables for these statement forms are

p	q	$\sim q$	$p \to q$	$p \vee \sim q$
T	T	F	T	T
T	F	T	F	T
F	T	F	T	F
F	F	T	T	T

Rows 2 and 3 of the table show that $p \to q$ and $p \lor \sim q$ do not always have the same truth values, and so $p \to q \not\equiv p \lor \sim q$.

Solution 2: The given statements are not logically equivalent. Let p be "Sam bought it at Crown Books," and q be "Sam paid full price." Then the two statements have the following form:

$$p \to \sim q \quad \text{and} \quad p \lor q.$$

The truth tables for these statement forms are

p	q	$\sim q$	$p \to \sim q$	$p \lor q$
T	T	F	F	T
T	F	T	T	T
F	T	F	T	T
F	F	T	T	F

Rows 1 and 4 of the table show that $p \to \sim q$ and $p \lor q$ do not always have the same truth values, and so $p \to \sim q \not\equiv p \lor q$.

7. The given statements are not logically equivalent. Let p be "Sam is out of Schlitz," and q be "Sam is out of beer." Then the two statements have the following form:

$$p \to q \quad \text{and} \quad \sim q \lor \sim p.$$

The truth tables for these statement forms are

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \lor \sim q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	T	T	T

The table shows that $p \to q$ and $\sim p \lor \sim q$ sometimes have opposite truth values (shown in rows 1 and 2), and so $p \to q \not\equiv \sim p \lor \sim q$.

8. *Converse*: If Jose is Jan's cousin, then Ann is Jan's mother

   Inverse: If Ann is not Jan's mother, then Jose is not Jan's cousin.

   Contrapositive: If Jose is not Jan's cousin, then Ann is not Jan's mother.

9. *Converse*: If Liu is Sue's cousin, then Ed is Sue's father.

   Inverse: If Ed is not Sue's father, then Liu is not Sue's cousin

   Contrapositive: If Liu is not Sue's cousin, then Ed is not Sue's father.

10. *Converse*: If Jim is Tom's grandfather, then then Al is Tom's cousin.

    Inverse: If Al is not Tom's cousin, then Jim is not Tom's grandfather

    Contrapositive: If Jim is not Tom's grandfather, then then Al is not Tom's cousin.

11. If someone does not get an answer of 10 for problem 16, then the person will not have solved problem 16 correctly.

    Or: If someone solves problem 16 correctly, then the person got an answer of 10.

12. *Sample answers:*

    For a form of argument to be valid means that no matter what statements are substituted for its statement variables, if the resulting premises are all true, then the conclusion is also true.

    Or: For a form of argument to be valid means that no matter what statements are substituted for its statement variables, it is impossible for all the premises to be true at the same time that the conclusion is false.

    Or: For a form of argument to be valid means that no matter what statements are substituted for its statement variables, it is impossible for conclusion to be false if all the premises are true.

13. The given form of argument is invalid.

p	q	$\sim p$	$\sim q$	$p \to \sim q$ (premises)	$q \to \sim p$ (premises)	$p \vee q$ (conclusion)
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	F

Row 4 of the truth table shows that it is possible for an argument of this form to have true premises and a false conclusion.

14. The given form of argument is invalid.

p	q	r	$\sim q$	$p \wedge \sim q$	$p \wedge \sim q \to r$ (premises)	$p \vee q$ (premises)	$q \to p$ (premises)	r (conclusion)
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	F
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	F	T	T	F	T
F	T	F	F	F	T	T	F	F
F	F	T	T	F	T	F	T	T
F	F	F	T	F	T	F	T	F

Row 2 of the truth table shows that it is possible for an argument of this form to have true premises and a false conclusion.

15. Let p be "Hugo is a physics major," q be "Hugo is a math major," and r be "Hugo needs to take calculus." Then the given argument has the following form:

$$\begin{array}{ll} & p \vee q \to r \\ & r \vee q \\ \text{Therefore} & p \vee q. \end{array}$$

Truth table:

p	q	r	$p \vee q$	$p \vee q \to r$ (premises)	$r \vee q$ (premises)	$p \vee q$ (conclusion)
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	F	T	T	F
F	F	F	F	T	F	F

Row 7 of the truth table shows that it is possible for an argument of this form to have true premises and a false conclusion. Therefore, the given argument is invalid.

16. Let p be "12 divides 709,438," q be "3 divides 709,438," and r be "The sum of the digits of 709,438 is divisible by 9." Then the given argument has the following form:

$$\begin{array}{ll} & p \to q \\ & r \to q \\ & \sim r \\ \text{Therefore} & \sim p. \end{array}$$

Truth table:

p	q	r	$\sim q$	$p \wedge \sim q$	$p \to q$ (premises)	$r \to q$ (premises)	$\sim r$ (premises)	$\sim p$ (conclusion)
T	T	T	F	F	T	T	F	F
T	T	F	F	F	T	T	T	F
T	F	T	T	T	F	F	F	F
T	F	F	T	T	F	T	T	F
F	T	T	F	F	T	T	F	T
F	T	F	F	F	T	T	T	T
F	F	T	T	F	T	F	F	T
F	F	F	T	F	T	T	T	T

Row 2 of the truth table shows that it is possible for an argument of this form to have true premises and a false conclusion. Therefore, the given argument is invalid.

17. The argument has the form

$$\begin{array}{rl} & p \to q \\ & \sim q \\ \text{Therefore} & \sim p, \end{array}$$

which is valid by modus tollens (and the fact that the negation of "17 is not a divisor of 54,587" is "17 is a divisor of 54,587").

18. The argument has the form

$$\begin{array}{rl} & p \to q \\ & q \\ \text{Therefore} & p, \end{array}$$

which is invalid; it exhibits the converse error.

19. A and B are knights, and C is a knave.

Reasoning: A cannot be a knave because if A were a knave his statement would be true, which is impossible for a knave. Hence A is a knight, and at least one of the three is a knave. That implies that at most two of the three are knaves, which means that B's statement is true. Hence B is a knight. Since at least one of the three is a knave and both A and B are knights, it follows that C is a knave.

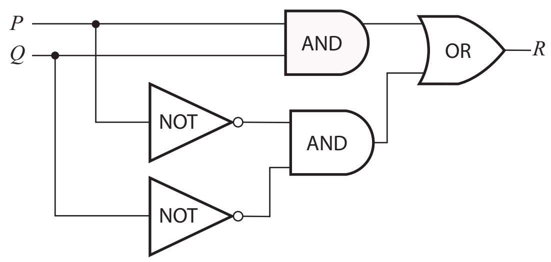
20. a. $S = 1$

b. $\sim (P \wedge Q) \wedge (Q \wedge R)$

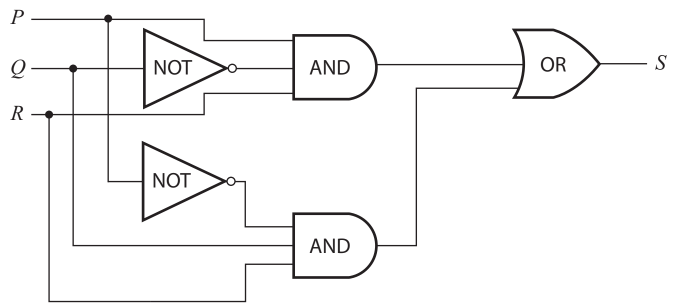
21. $110101_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^0 = 32 + 16 + 4 + 1 = 53_{10}$

22. $75_{10} = 64 + 8 + 2 + 1 = 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 1001011_2$.

23. The following circuit corresponds to the given Boolean expression:



24. One circuit (among many) having the given input/output table is the following:



25.

$$\begin{array}{rr} & 10111_2 \\ + & 1011_2 \\ \hline & 100010_2 \end{array}$$

26. $100110_2 = 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 32 + 4 + 2 = 38_{10}$

27. $49_{10} = (32 + 16 + 1)_{10} = 00110001_2 \longrightarrow 11001110 \longrightarrow 11001111.$

    So the two's complement is 11001111.

    Check: $2^8 - 49 = 256 - 49 = 207$ and

$$\begin{aligned} 11001111_2 &= 1 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 128 + 64 + 8 + 4 + 2 + 1 \\ &= 207, \end{aligned}$$

    which agrees.

Chapter 3

1. $\forall$ valid argument x, if x has true premises, then x has a true conclusion.

2. a. $\forall$ odd integer n, n^2 is odd.

   b. $\forall$ integer n, if n is odd then n^2 is odd.

   c. $\exists$ an odd integer n such that n^2 is not odd.

   Or: $\exists$ an integer n such that n is odd and n^2 is not odd.

3. $\forall$ rational number r, $\exists$ integers u and v such that r is the ratio of u to v.

   Or: $\forall$ rational number r, $\exists$ integers u and v such that $r = u/v$.

4. $\forall$ even integer n that is greater than 2, $\exists$ prime numbers p and q such that $n = p + q$.

   Or: $\forall$ even integer n, if $n > 2$ then $\exists$ prime numbers p and q such that $n = p + q$.

5. e

6. a

7. d

8. g

9. a. There is an integer n such that n is prime and n is not odd.

   b. $\exists$ a real number x such that $x < 1$ and $\frac{1}{x} \not> 1$.

   Or: $\exists$ a real number x such that $x < 1$ and $\frac{1}{x} \leq 1$.

   c. There are integers a and b such that a^2 divides b^2 and a does not divide b.

   d. $\exists$ a real number x such that $x(x-2) > 0$ and $x \not> 2$ and $x \not< 0$.

   Or: $\exists$ a real number x such that $x(x-2) > 0$ and $0 \leq x \leq 2$.

   e. $\exists$ a real number x such that $x(x-2) > 0$ and either $0 > x$ or $x > 2$.

   Or: $\exists$ a real number x such that $x(x-2) > 0$ and either $0 \not\leq x$ or $x \not\leq 2$.

   f. There are real numbers x and y with $x < y$ such that for all integers n, either $x > n$ or $n > y$.

   Or: There are real numbers x and y with $x < y$ such that for all integers n, either $x \not\leq n$ or $n \not\leq y$.

10. a. $\forall$ real numbers x, if $x+1 > 0$ then $-1 < x \leq 0$.

    b. $\forall$ real numbers x, if $x+1 \leq 0$ then $-1 \geq x$ or $x > 0$.

    Or: $\forall$ real numbers x, if $x+1 \not> 0$ then $-1 \not< x$ or $x \not\leq 0$.

11. If a graph with n vertices is a tree, then it has $n-1$ edges.

12. These two statements are not logically equivalent.

    Explanation 1: The first statement is equivalent to "If a real number is less than 1, then its reciprocal is greater than 1" and the second statement is equivalent to "If the reciprocal of a real number is greater than 1, then the number is less than 1." Thus the second statement is the converse of the first, and a conditional statement and its converse are not logically equivalent.

    Explanation 2: The first statment is false. For example, -2 is less than 1, but its reciprocal, $-\frac{1}{2}$ is greater than 1. However, the second statement is true; if the reciprocal of a real number is greater than 1, then the number itself is positive and is between 0 and 1, and it is impossible for one of a pair of equivalent statements to be true while the other member of the pair is false.

13. a. For any integer you might choose, you can find an integer such that the sum of the two equals 0.

    Or: Every integer has an additive inverse.

    b. There is an integer with the property that no matter what integer is chosen, the sum of the two will be 0.

    Or: Some integer has the property that its sum with every integer equals 0.

14. a. Given any real number, there is a real number that is less than the given number.

    Or: There is no smallest real number.

    b. There is a real number that is less than every real number.

15. The argument is valid by modus tollens.

16. The argument is invalid; it exhibits the inverse error.

Chapter 4

1. *Some acceptable answers*:

   a. An integer n is odd if, and only if, $n = 2k+1$ for some integer k.

   b. An integer n is odd if, and only if, n equals 2 times some integer plus 1.

   c. An integer n is odd if, and only if, there exists an integer m such that $n = 2m+1$.

2. *Counterexample:* Let $a = 1$, $b = 2$, $c = 1$, and $d = 2$. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{1}{2} + \frac{1}{2} = 1, \text{ whereas } \frac{a+c}{b+d} = \frac{1+1}{2+2} = \frac{2}{4} = \frac{1}{2}.$$

(This is one counterexample among many.)

3. a. $\exists$ integers m and n such that $2m + n$ is odd and m and n are not both odd. In other words, $\exists$ integers m and n such that $2m + n$ is odd and at least one of m and n is even.

   b. Statement A can be shown to be false by giving a counterexample.

   Counterexample: Let $m = 1$ and $n = 2$. Then $2m + n = 2 \cdot 1 + 2 = 4$, which is even, but it is not the case that both m and n are odd because n is even.(This is one counterexample among many.)

4. No matter what integers replace m and n in the expression $6m^2 + 34n - 18$, the result will always be an even integer. To see this, observe that

$$6m^2 + 34n - 18 \quad = \quad 2(m^2 + 17n - 9).$$

   Since products, sums, and differences of integers are integers, $m^2 + 17n - 9$ is an integer. Thus $6m^2 + 34n - 18$ can be written as 2 times an integer, and so it is an even integer.

5. *Counterexample:* Let $a = b = \sqrt{2}$. Then a and b are irrational, and $ab = \sqrt{2} \cdot \sqrt{2} = 2$, which is rational.

6. *Some acceptable answers*:

   a. A real number r is rational if, and only if, $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

   b. A real number r is rational if, and only if, there exist integers a and b such that $r = \frac{a}{b}$ and $b \neq 0$.

   c. A real number r is rational if, and only if, r can be written as a ratio of integers with a nonzero denominator.

7. 605.83 is a rational number because $605.83 = \dfrac{60583}{100}$.

8. 56.745 a rational number because $56.745 = \dfrac{56745}{1000}$.

9. *Some acceptable answers*:

   a.An integer n is divisible by an integer d if, and only if, $n = dk$ for some integer k.

   b. An integer n is divisible by an integer d if, and only if, there exists an integer k so that $n = dk$.

   c. An integer n is divisible by an integer d if, and only if, n is equal to d times some integer.

10. 0 is divisible by 3 because $0 = 3 \cdot 0$.

11. 12 divides 72 because $72 = 12 \cdot 6$.

12. Proof: Suppose r and s, are any *[particular but arbitarily chosen]* rational numbers.
    We must show that $r - s$ is rational.

13. Proof: Suppose r and s, are any rational numbers with $s \neq 0$. *[We must show that* $\dfrac{2r}{5s}$ *is a rational number.]*By definition of rational, there exist integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ such that

$$r = \frac{a}{b} \quad \text{and} \quad s = \frac{c}{d}.$$

    In addition, since $c = s \cdot d$ and since neither s nor d equals 0, then $c \neq 0$ by the zero product property. Then

$$\frac{2r}{5s} = \frac{2 \cdot \frac{a}{b}}{5 \cdot \frac{c}{d}} = \frac{2ad}{5bc}.$$

    Now both $2ad$ and $5bc$ are integers because they are products of integers, and $5bc \neq 0$ by the zero product property. Hence $\dfrac{2r}{5s}$ is a ratio of integers with a nonzero denominator, and so $\dfrac{2r}{5s}$ is a rational number by definition of rational *[as was to be shown]*.

14. Proof : Suppose a, b, and c are any integers such that $a \mid b$ and $a \mid c$. *[We must show that $a \mid (5b+3c)$.]* By definition of divisibility, $b = ar$ and $c = as$ for some integers r and s. Then

$$\begin{aligned} 5b + 3c &= 5(ar) + 3(as) \quad \text{by substitution} \\ &= a(5r + 3s) \quad \text{by the commutative and associative laws of algebra.} \end{aligned}$$

Let $t = 5r+3s$. Then t is an integer because products and sums of integers are integers, and $5b+3c = at$. Thus, by definition of divisibility, $a \mid (5b + 3c)$ *[as was to be shown]*.

Prove the following statement directly from the definitions of the terms. Do not use any other facts previously proved in class or in the text or in the exercises.

For all integers a, b, and c, if $a \mid b$ and $a \mid c$, then $a \mid (5b + 3c)$.

15. Proof *(Version 1)*: Suppose a and b are integers and a divides b. By definition of divisibility, $b = a \cdot k$ for some integer k. Raising both sides to the third power gives

$$\begin{aligned} b^3 &= (a \cdot k)^3 \\ &= a^3 \cdot k^3 \quad \text{by the commutative and associative laws of algebra.} \end{aligned}$$

Observe that k^3 is an integer because it is a product of integers. Hence b^3 equals a^3 times an integer, and so by definition of divisibility, a^3 divides b^3.

Proof *(Version 2)*: Suppose a and b are integers and a divides b. By definition of divisibility, $b = a \cdot k$ for some integer k. Raising both sides to the third power gives

$$\begin{aligned} b^3 &= (a \cdot k)^3 \\ &= a^3 \cdot k^3 \quad \text{by the commutative and associative laws of algebra.} \end{aligned}$$

Let $t = k^3$. Then t is an integer because it is a product of integers. Hence $b^3 = a^3 \cdot t$, where t is an integer, and so by definition of divisibility, a^3 divides b^3.

16. Proof: Suppose n is any integer. By the parity principle, either n is even or n is odd, so we consider two cases.

Case 1 (n is even): By definition of even, there is an integer u such that $n = 2u$. Then

$$n^2 + n + 1 = (2u)^2 + 2u + 1 = 4u^2 + 2u + 1 = 2(2u^2 + u) + 1.$$

Now $2u^2 + u$ is an integer because products and sums of integers are integers. Thus $n^2 + n + 1$ equals 2 times an integer plus 1, and so $n^2 + n + 1$ is odd by definition of odd.

Case 2 (n is odd): By definition of odd, there is an integer u such that $n = 2u + 1$. Then

$$n^2+n+1 = (2u+1)^2+(2u+1)+1 = 4u^2+4u+1+2u+1+1 = 4u^2+6u+2+1 = 2(2u^2+u+1)+1.$$

Now $2u^2 + u + 1$ is an integer because products and sums of integers are integers. Thus $n^2 + n + 1$ equals 2 times an integer plus 1, and so $n^2 + n + 1$ is odd by definition of odd.

Conclusion: In both possible cases $n^2 + n + 1$ is odd.

17. Proof: Suppose n is any integer. By the parity principle, either n is even or n is odd, so we consider two cases.

Case 1 (n is even): By definition of even, there is an integer a such that $n = 2a$. The next consecutive integer after n is $n + 1$, and so

$$n + (n + 1) = 2a + (2a + 1) = 4a + 1 = 2(2a) + 1.$$

Now $2a$ is an integer because products of integers are integers. Thus $n + (n + 1)$ equals 2 times an integer plus 1, and so $n + (n + 1)$ is odd by definition of odd.

Case 2 (n is odd): By definition of odd, there is an integer a such that $n = 2a + 1$. Then

$$n + (n + 1) = (2a + 1) + [(2a + 1) + 1] = 4a + 2 + 1 = 2(2a + 1) + 1.$$

Now $2a + 1$ is an integer because products and sums of integers are integers. Thus $n + (n + 1)$ equals 2 times an integer plus 1, and so $n + (n + 1)$ is odd by definition of odd.

Conclusion: In both possible cases $n + (n + 1)$ is odd.

Prove the following statement: The sum of any two consecutive integers can be written in the form $4n + 1$ for some integer n.

18. Proof: Suppose x is any real number. By definition of floor,

$$\lfloor x - 2 \rfloor = n \quad \text{if,and only if,} \quad n \le x - 2 < n + 1.$$

Adding 2 to all parts of the inequality gives

$$n + 2 \le x < n + 3.$$

Thus, by definition of floor,

$$\lfloor x \rfloor = n + 2, \quad \text{and so} \quad \lfloor x \rfloor - 2 = n.$$

Thus both $\lfloor x - 2 \rfloor$ and $\lfloor x \rfloor - 2$ equal the same quantity (namely n), and so $\lfloor x - 2 \rfloor = \lfloor x \rfloor - 2$.

19. Proof (by contradiction): Suppose not. That is, suppose there is a smallest positive rational number; call it r. *[We must show that this supposition leads logically to a contradiction.]* By definition of rational, there exist integers u and v with $v \neq 0$ such that $r = \dfrac{u}{v}$. Then

$$\frac{r}{2} = \frac{\frac{u}{v}}{2} = \frac{u}{2v}.$$

Now both u and $2v$ are integers because products of integers are integers, and $2v \neq 0$ by the zero product property. Thus $\dfrac{r}{2}$ is a raional number.

In addition, since r is positive, $0 < r$. Hence, by adding r to both sides, $r < 2r$, and dividing both sides by 2 gives that $\dfrac{r}{2} < r$, so $\dfrac{r}{2}$ is less than r.

Moreover, dividing both sides of $0 < r$ by 2 gives that $0 < \dfrac{r}{2}$, so $\dfrac{r}{2}$ is positive.

In summary: $\dfrac{r}{2}$ is a positive rational number that is less than r, which contradicts the supposition that r is the smallest positive rational number. *[Hence the supposition is false and the given statement is true.]*

20. Proof (by contradiction):Suppose not. That is, suppose there are real numbers r and s such that r is rational and s is irrational and $r + 2s$ is rational. *[We must show that this supposition leads logically to a contradiction.]* By definition of rational,

$$r = \frac{a}{b} \quad \text{and} \quad r + 2s = \frac{c}{d} \quad \text{for some integers } a,\ b,\ c, \text{ and } d \text{ with } b \neq 0 \text{ and } d \neq 0.$$

Then, by substitution,

$$\frac{a}{b} + 2s = \frac{c}{d}.$$

Solve this equation for s to obtain

$$s = \frac{1}{2}\left(\frac{c}{d} - \frac{a}{b}\right) = \frac{1}{2}\left(\frac{bc}{bd} - \frac{ad}{bd}\right) = \frac{bc - ad}{2bd}.$$

But both $bc - ad$ and $2bd$ are integers because products and differences of integers are integers, and $2bd \neq 0$ by the zero product property.

Hence s is a ratio of integers with a nonzero denominator, and so s is rational by definition of rational.

This contradicts the supposition that x is irrational. *[Hence the supposition is false and the given statement is true.]*

21. *Solution 1:*

a. Proof (by contradiction): Suppose not. That is, suppose there is an integer n such that n^3 is even and n is odd.

[We must show that this supposition leads logically to a contradiction.]

By definition of odd, $n = 2a + 1$ for some integer a.

Thus, by substitution and algebra,

$$n^3 = (2a+1)^3 = (2a+1)^2(2a+1) = (4a^2+4a+1)(2a+1) = 8a^3+12a^2+6a+1 = 2(4a^3+6a^2+3a)+1.$$

Let $t = 4a^3+6a^2+3a$. Then $n^3 = 2t+1$, and t is an integer because it is a sum of products of integers.

It follows that n^3 is odd by definition of odd, which contradicts the supposition that n^3 is even.

[Hence the supposition is false and the given statement is true.]

b. Outline of proof by contraposition:

Starting point: Suppose n is any integer such that n is odd. (*Or:* Suppose n is any odd integer.)

Conclusion to be shown: n^3 is odd.

Solution 2:

a. Proof (by contraposition): Suppose n is any integer such that n is odd. (*Or:* Suppose n is any odd integer.) *[We must show that n^3 is odd.]*

By definition of odd, $n = 2a + 1$ for some integer a.

Thus $n^3 = (2a+1)^3 = (2a+1)^2(2a+1) = (4a^2+4a+1)(2a+1) = 8a^3+12a^2+6a+1 = 2(4a^3+6a^2+3a)+1$.

Let $t = 4a^3+6a^2+3a$. Then $n^3 = 2t+1$, and t is an integer because it is a sum of products of integers.

It follows that n^3 is odd by definition of odd *[as was to be shown]*.

b. Outline of proof by contradiction:

Starting point: Suppose not. That is, suppose there is an integer n such that n^3 is even and n is odd.

Conclusion to be shown: We must show that this supposition leads logically to a contradiction.

22. *Solution 1:*

a. Proof (by contradiction): Suppose not. That is, suppose there is a real number r such that r^3 is irrational and r is rational.

[We must show that this supposition leads logically to a contradiction.]

By definition of rational, there exist integers a and b with $b \neq 0$ such that $r = \dfrac{a}{b}$.

By substitution, $r^3 = \left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$.

Now both a^3 and b^3 are integers because products of integers are integers, and $b^3 \neq 0$ by the zero product property.

Thus r^3 is a ratio of integers with a nonzero denominator, and so r^3 is rational by definition of rational.

This contradicts the supposition that r^3 is irrational. *[Hence the supposition is false and the given statement is true.]*

b. Outline of proof by contraposition:

Starting point: Suppose r is any real number such that r is rational. (*Or:* Suppose r is any rational number.)

Conclusion to be shown: r^3 is rational.

Solution 2:

a. Proof (by contraposition): Suppose r is any real number such that r is rational. (*Or:* Suppose r is any rational number.)

[We must show that r^3 is rational.]

By definition of rational, there exist integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$.

By substitution, $r^3 = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$.

Now both a^3 and b^3 are integers because products of integers are integers, and $b^3 \neq 0$ by the zero product property.

Thus r^3 is a ratio of integers with a nonzero denominator, and so r^3 is rational by definition of rational *[as was to be shown]*.

b. Outline of proof by contradiction:

Starting point: Suppose not. That is, suppose there is a real number r such that r^3 is irrational and r is rational.

Conclusion to be shown: We must show that this supposition leads logically to a contradiction.

23. *Solution 1:*

a. Proof (by contradiction): Suppose not. That is, suppose there is an integer n such that n^3 is odd and n is even.

[We must show that this supposition leads logically to a contradiction.]

By definition of even, there is an integer a such that $n = 2a$.

Thus, by substitution and algebra,

$$n^3 = (2a)^3 = 8a^3 = 2(4a^3).$$

Let $t = 4a^3$. Then $n^3 = 2t$, and t is an integer because it is a product of integers.

It follows that n^3 is even by definition of even, which contradicts the supposition that n^3 is odd.

[Hence the supposition is false and the given statement is true.]

b. Outline of proof by contraposition:

Starting point: Suppose n is any integer such that n is even. (*Or:* Suppose n is any even integer.)

Conclusion to be shown: n^3 is even.

Solution 2:

a. Proof (by contraposition): Suppose n is any integer such that n is even. (*Or:* Suppose n is any even integer.) *[We must show that n^3 is even.]*

By definition of even, there is an integer a such that $n = 2a$.

Thus, by substitution and algebra,

$$n^3 = (2a)^3 = 8a^3 = 2(4a^3).$$

Let $t = 4a^3$. Then $n^3 = 2t$, and t is an integer because it is a product of integers.

It follows that n^3 is even by definition of even *[as was to be shown]*.

b. Outline of proof by contradiction:

Starting point: Suppose not. That is, suppose there is an integer n such that n^3 is odd and n is even.

Conclusion to be shown: We must show that this supposition leads logically to a contradiction.

24. The given statement is false.

Counterexample: Let $r = \sqrt{2}$. Then r is irrational, but $r^2 = (\sqrt{2})^2 = 2$ is rational.

25. (i). rational

(ii). a and b are integers and $b \neq 0$

(iii). $a - 7b$

(iv). $\frac{a - 7b}{4b}$

(v). both $a - 7b$ and $4b$ are integers (since products and differences of integers are integers) and so $\sqrt{2}$ would be a rational number

(vi). the supposition is false and the given statement is true.

26. <u>Proof</u>: Suppose not. That is, suppose that $4+3\sqrt{2}$ is rational. By definition of rational, $7+4\sqrt{2} = \frac{a}{b}$, where a and b are integers and $b \neq 0$. Multiplying both sides by b gives

$$4b + 3b\sqrt{2} = a,$$

so if we subtract $4b$ from both sides we have

$$3b\sqrt{2} = a - 4b.$$

Dividing both sides by $3b$ gives

$$\sqrt{2} = \frac{a-4b}{3b}.$$

But then $\sqrt{2}$ would be a rational number because both $a-4b$ and $3b$ are integers (since products and differences of integers are integers) and so $\sqrt{2}$ would be a rational number. This contradicts our knowledge that $\sqrt{2}$ is irrational. Hence the supposition is false and the given statement is true.

27.

$$\begin{array}{r} 1 \\ 168\overline{)284} \\ \underline{168} \\ 116 \end{array}$$

So $284 = 168 \cdot 1 + 116$, and hence $\gcd(284, 168) = \gcd(168, 116)$

$$\begin{array}{r} 1 \\ 116\overline{)168} \\ \underline{116} \\ 52 \end{array}$$

So $168 = 116 \cdot 1 + 52$, and hence $\gcd(168, 116) = \gcd(116, 52)$

$$\begin{array}{r} 2 \\ 52\overline{)116} \\ \underline{104} \\ 12 \end{array}$$

So $116 = 52 \cdot 2 + 12$, and hence $\gcd(116, 52) = \gcd(52, 12)$

$$\begin{array}{r} 4 \\ 12\overline{)52} \\ \underline{48} \\ 4 \end{array}$$

So $52 = 12 \cdot 4 + 4$, and hence $\gcd(52, 12) = \gcd(12, 4)$

$$\begin{array}{r} 3 \\ 4\overline{)12} \\ \underline{12} \\ 0 \end{array}$$

So $12 = 4 \cdot 3 + 0$, and hence $\gcd(12, 4) = \gcd(4, 0)$

But $\gcd(4, 0) = 4$. So $\gcd(284, 168) = 4$.

28.

1
10673 | 11284 — So $11284 = 10673 \cdot 1 + 611$, and hence $\gcd(11284, 10673) = \gcd(10673, 611)$
$\underline{10673}$
611

17
611 | 10673 — So $10673 = 611 \cdot 17 + 286$, and hence $\gcd(10673, 611) = \gcd(611, 286)$
$\underline{10387}$
286

2
286 | 611 — So $611 = 52 \cdot 2 + 12$, and hence $\gcd(611, 286) = \gcd(286, 39)$
$\underline{572}$
39

7
39 | 286 — So $286 = 39 \cdot 4 + 4$, and hence $\gcd(286, 39) = \gcd(39, 13)$
$\underline{273}$
13

3
13 | 39 — So $39 = 13 \cdot 3 + 0$, and hence $\gcd(39, 13) = \gcd(13, 0)$
$\underline{39}$
0

But $\gcd(13, 0) = 13$. So $\gcd(11284, 10673) = 13$.